Foundations of Quantum Mechanics

States, Measurement, and the Quantum Harmonic Oscillator

Purpose: Introduce quantum states, superposition, probabilistic measurement, and interference, and use the quantum harmonic oscillator to develop raising and lowering operators, number states, and zero-point energy.

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Prerequisites. Complex numbers, basic linear algebra, introductory classical mechanics, and familiarity with waves and interference.

Learning objectives. After completing these notes, the reader should be able to:

- interpret kets, bras, and inner products;
- explain superposition and probabilistic measurement;
- distinguish coherent quantum interference from a classical mixture;
- derive the oscillator Hamiltonian using ladder operators;
- construct number states and identify zero-point energy.

1 Introduction

Quantum mechanics requires a different descriptive language from classical mechanics. Physical states are represented by vectors, measurable outcomes are probabilistic, and distinct alternatives can combine through complex probability amplitudes. The double-slit experiment provides a direct illustration: individual detections are localized, yet repeated measurements reveal an interference pattern that cannot be explained by adding classical probabilities alone.

The quantum harmonic oscillator provides an ideal setting in which to develop the operator structure of quantum mechanics. By introducing raising and lowering operators, the Hamiltonian can be rewritten in a form that reveals the energy spectrum, the number-state basis, and the origin of zero-point energy without first solving the position-space differential equation.

The emphasis throughout is conceptual. The mathematics is introduced to make the physical reasoning precise and to show how a small set of principles leads to experimentally testable predictions.

2 Quantum states and superposition

Quantum mechanics describes physical systems using *state vectors*. In Dirac notation, a state vector is written as a *ket*. For a two-state system, we choose the basis kets

$$|0\rangle\,, \qquad |1\rangle\,. \tag{1}$$

The symbol $|0\rangle$, read “ket zero,” denotes a vector in the state space; similarly, $|1\rangle$ is read “ket one.”

2.1 From kets to bras

The Hermitian conjugate of a ket is called a *bra*. Thus,

$$\langle 0| = (|0\rangle)^\dagger\,, \qquad \langle 1| = (|1\rangle)^\dagger\,. \tag{2}$$

More generally, if

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle , \tag{3}$$

then taking the Hermitian conjugate gives

$$\langle\psi| = (|\psi\rangle)^\dagger = a_0^* \langle 0| + a_1^* \langle 1| . \tag{4}$$

The coefficients are complex conjugated because Hermitian conjugation includes complex conjugation.

A bra acting on a ket produces an *inner product*, which is a complex number:

$$\langle\phi|\psi\rangle . \tag{5}$$

For an orthonormal basis,

$$\langle i|j\rangle = \delta_{ij}. \tag{6}$$

Thus, the basis states are normalized and mutually orthogonal:

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0. \tag{7}$$

> A ket is a vector, its Hermitian conjugate is a bra, and a bra–ket combination such as $\langle\phi|\psi\rangle$ is an inner product.

2.2 Superposition and normalization

The superposition principle states that a general pure state can be written as

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle , \tag{8}$$

where $a_0, a_1 \in \mathbb{C}$ are probability amplitudes.

A physical state must be normalized:

$$\begin{aligned} 1 &= \langle\psi|\psi\rangle && (9)\\ &= (a_0^* \langle 0| + a_1^* \langle 1|)(a_0 |0\rangle + a_1 |1\rangle) && (10)\\ &= |a_0|^2 \langle 0|0\rangle + a_0^* a_1 \langle 0|1\rangle + a_1^* a_0 \langle 1|0\rangle + |a_1|^2 \langle 1|1\rangle && (11)\\ &= |a_0|^2 + |a_1|^2. && (12) \end{aligned}$$

The cross terms vanish because the basis states are orthogonal.

2.3 Born interpretation

If the state $|\psi\rangle$ is measured in the basis $\{|0\rangle , |1\rangle\}$, the possible outcomes and their probabilities are

$$P(0) = |a_0|^2, \qquad P(1) = |a_1|^2. \tag{13}$$

Because the state is normalized,

$$P(0) + P(1) = 1. \tag{14}$$

> Probabilities are obtained by taking the absolute square of probability amplitudes.

Consider the state

$$|\psi\rangle = c\,(|0\rangle + 2i\,|1\rangle) . \tag{15}$$

Its bra is

$$\langle\psi| = (|\psi\rangle)^\dagger = c^* \,(\langle 0| - 2i\,\langle 1|) . \tag{16}$$

Normalization gives

$$\begin{aligned} 1 &= \langle\psi|\psi\rangle && (17)\\ &= |c|^2 (1 + 4) = 5|c|^2. && (18) \end{aligned}$$

Therefore,

$$|c| = \frac{1}{\sqrt{5}}. \tag{19}$$

A convenient phase convention is to choose

$$c = \frac{1}{\sqrt{5}}, \tag{20}$$

so that

$$|\psi\rangle = \frac{1}{\sqrt{5}} \left(|0\rangle + 2i\,|1\rangle \right). \tag{21}$$

The measurement probabilities are then

$$P(0) = \frac{1}{5}, \qquad P(1) = \frac{4}{5}. \tag{22}$$

3 The double-slit experiment

The double-slit experiment provides one of the clearest demonstrations that quantum mechanics requires a description based on probability amplitudes rather than definite classical trajectories. We begin with the observations before introducing the mathematical explanation.

3.1 What is observed?

A source emits photons toward a barrier containing two narrow slits. A detection screen is placed behind the barrier. The experiment is performed at sufficiently low intensity that, to a very good approximation, only one photon is present in the apparatus at a time.

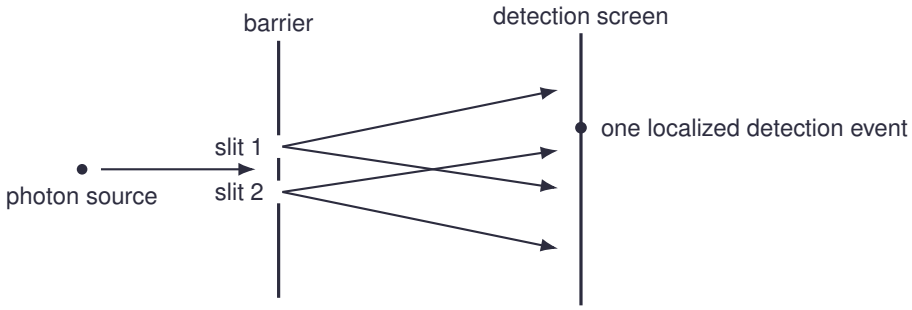

Figure 1: A double-slit experiment operated in the single-photon regime. Each photon is detected at one localized position on the screen. The arrows are schematic and do not represent measured classical trajectories.

Each experimental trial produces a single localized event: the detector records one photon at one position. A single detection does not reveal an interference pattern. The pattern appears only after the experiment has been repeated many times under the same preparation conditions.

Each repetition again produces one localized event, but the accumulated spatial distribution is not uniform: regions of high detection probability alternate with regions of low detection probability. These maxima and minima form the interference pattern.

A single event therefore cannot be identified as a “piece” of a fringe. The fringe pattern is a statistical property of an ensemble of identically prepared trials.

3.2 The conceptual lesson

The experiment separates two ideas that are easily confused:

1. Each photon is detected as a localized event.
2. The probability distribution governing many such events displays interference.

Before detection, quantum mechanics does not assign the photon a measured position on the screen or a directly observed classical path through one particular slit. Instead, the preparation and the apparatus determine a probability amplitude for every possible detection position. The uncertainty is therefore not merely ignorance about an already measured position; it is part of the quantum description prior to detection.

A single photon produces a single localized detection event, but many identically prepared single-photon trials build an interference pattern. Particle-like detection and wave-like probability amplitudes are both required in the description.

Concept before calculation. The double-slit experiment is not difficult because its mathematics is complicated. Its difficulty is conceptual: alternatives are combined at the level of probability amplitudes, and only afterward are measurable probabilities obtained. The mathematical derivation in the next section makes this statement precise.

3.3 Ideal interference and visibility

When the two alternatives remain fully coherent and no information is available that distinguishes the slit taken by the photon, the interference contrast is maximal. This ideal case is described by visibility

$$V = 1. \tag{23}$$

At this stage, visibility is introduced as an experimental characterization of the contrast between bright and dark fringes. Its mathematical definition and the amplitude calculation will follow after the conceptual picture is established.

4 Probability amplitudes and interference

The conceptual discussion can now be expressed mathematically. Let $\psi_1(x)$ and $\psi_2(x)$ denote the probability amplitudes for a photon to be detected at position x on the screen through the two indistinguishable alternatives associated with slit 1 and slit 2. The superposition principle states that the total probability amplitude is

$$\psi(x) = \psi_1(x) + \psi_2(x). \tag{24}$$

The amplitudes are added first. The measurable detection probability density is obtained only afterward by applying the Born rule:

$$P(x) = |\psi(x)|^2. \tag{25}$$

Thus,

$$\begin{aligned} P(x) &= |\psi_1(x) + \psi_2(x)|^2 && (26)\\ &= (\psi_1^*(x) + \psi_2^*(x))\,(\psi_1(x) + \psi_2(x)) && (27)\\ &= |\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1^*(x)\psi_2(x) + \psi_2^*(x)\psi_1(x) && (28)\\ &= |\psi_1(x)|^2 + |\psi_2(x)|^2 + 2\,\mathrm{Re}[\psi_1^*(x)\psi_2(x)]\,. && (29) \end{aligned}$$

The first two terms are the individual single-slit contributions. The final term is the interference contribution.

To make the phase dependence explicit, write

$$\psi_j(x) = A_j(x)e^{i\phi_j(x)}, \qquad j \in \{1, 2\}, \tag{30}$$

where $A_j(x) \geq 0$. Then

$$P(x) = A_1^2(x) + A_2^2(x) + 2A_1(x)A_2(x)\cos[\phi_2(x) - \phi_1(x)]\,. \tag{31}$$

Constructive and destructive interference arise from the relative phase $\phi_2(x) - \phi_1(x)$, not from either alternative by itself.

4.1 Visibility as an interference control

The visibility parameter introduced above can be used to control the strength of the phase-sensitive interference term:

$$P_V(x) = A_1^2(x) + A_2^2(x) + 2VA_1(x)A_2(x)\cos[\phi_2(x) - \phi_1(x)]\,, \qquad 0 \leq V \leq 1. \tag{32}$$

For $V = 1$, the alternatives are fully coherent and the interference modulation is maximal. For $0 < V < 1$, the modulation is reduced. For $V = 0$, the interference term disappears and

$$P_{V=0}(x) = A_1^2(x) + A_2^2(x). \tag{33}$$

The probability density is therefore not zero: photons are still detected on the screen. What disappears is the structured fringe modulation produced by the cross term.

If the screen records an intensity or count-rate profile $I(x)$, the corresponding normalized position probability density is

$$p(x) = \frac{I(x)}{\int I(x')\,\mathrm{d}x'}, \tag{34}$$

so that $\int p(x)\,\mathrm{d}x = 1$.

4.2 Ensembles, uncertainty, and measurement

A single trial yields one localized outcome. The probability distribution is revealed only by repeating the same preparation and measurement many times. Quantum experiments are therefore interpreted through ensembles of repeated trials, even though each member of the ensemble is detected individually.

Before measurement, the photon is described by a quantum state that may be a coherent superposition of possible outcomes. This should not be interpreted as ordinary ignorance about a path that was already definite but unknown. The coherence between the alternatives has an observable consequence: it produces the interference term in $P_V(x)$. Measurement then produces one outcome, with probabilities determined by the quantum state rather than with certainty.

4.3 Conceptual boundary with classical statistical mechanics

In classical statistical mechanics, a probability distribution commonly represents incomplete knowledge about which definite microstate the system actually occupies. A classical mixture of the two slit alternatives would add probabilities,

$$P_{\mathrm{mix}}(x) = P_1(x) + P_2(x), \tag{35}$$

and would contain no phase-sensitive cross term. In the double-slit experiment, by contrast, the alternatives must be represented by coherent probability amplitudes before the absolute square is taken. The resulting interference term distinguishes a quantum superposition from an incoherent classical mixture.

> **Quantum uncertainty is not ordinary statistical ignorance.** A classical mixture assumes that the system occupies one definite alternative, even when that alternative is unknown. A coherent quantum superposition retains phase relations between alternatives, and those phase relations generate measurable interference terms.

5 The quantum harmonic oscillator

We now apply the operator formalism to one of the most important model systems in quantum mechanics: a particle of mass m moving in the quadratic potential

$$U(x) = \frac{1}{2}m\omega^2 x^2. \tag{36}$$

The Hamiltonian is the sum of kinetic and potential energy,

$$\hat{H} = \hat{K} + \hat{U} = \frac{\hat{p}^{\,2}}{2m} + \frac{1}{2}m\omega^2\hat{x}^{\,2}. \tag{37}$$

In the position representation, $\hat{x}$ acts by multiplication and $\hat{p} = -i\hbar\,\mathrm{d}/\mathrm{d}x$. Therefore,

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2. \tag{38}$$

5.1 Creation and annihilation operators

Instead of solving the differential equation directly, we introduce two new operators,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{39}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{40}$$

The operator $\hat{a}$ is called the *annihilation* or *lowering* operator, and $\hat{a}^\dagger$ is called the *creation* or *raising* operator. Because $\hat{x}$ and $\hat{p}$ are Hermitian, the two definitions are Hermitian conjugates of one another.

The canonical commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \tag{41}$$

implies

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{42}$$

To verify this explicitly, let $c = \sqrt{m\omega/(2\hbar)}$. Then

$$[\hat{a}, \hat{a}^\dagger] = c^2 \left[\hat{x} + \frac{i}{m\omega}\hat{p}, \hat{x} - \frac{i}{m\omega}\hat{p}\right] \tag{43}$$

$$= c^2 \left(-\frac{i}{m\omega}[\hat{x}, \hat{p}] + \frac{i}{m\omega}[\hat{p}, \hat{x}]\right) \tag{44}$$

$$= c^2 \frac{2\hbar}{m\omega} = 1. \tag{45}$$

5.2 Hamiltonian in ladder-operator form

Multiplying the two ladder operators gives

$$\hat{a}^\dagger \hat{a} = \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \tag{46}$$

$$= \frac{m\omega}{2\hbar}\hat{x}^{\,2} + \frac{1}{2m\hbar\omega}\hat{p}^{\,2} - \frac{1}{2}. \tag{47}$$

The Hamiltonian may first be written in the symmetric form

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger\right). \tag{48}$$

The commutator allows us to exchange the operator order:

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1, \tag{49}$$

so that

$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1. \tag{50}$$

Substituting this relation into the symmetric Hamiltonian gives

$$\hat{H} = \frac{\hbar\omega}{2}\left[\hat{a}^\dagger \hat{a} + \left(\hat{a}^\dagger \hat{a} + 1\right)\right] \tag{51}$$

$$= \frac{\hbar\omega}{2}\left(2\hat{a}^\dagger \hat{a} + 1\right) \tag{52}$$

$$= \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right). \tag{53}$$

The constant contribution $\hbar\omega/2$ anticipates the zero-point energy. It arises directly from the noncommutativity of the lowering and raising operators rather than being inserted by hand. On the next page, the ground-state condition $\hat{a}|0\rangle = 0$ will establish explicitly that the lowest oscillator energy is $E_0 = \hbar\omega/2$.

This algebraic form prepares us to determine the energy eigenstates without solving the position-space differential equation directly.

> **Origin of zero-point energy.** The term $\hbar\omega/2$ is generated by the noncommutativity of $\hat{a}$ and $\hat{a}^\dagger$. It is therefore a structural consequence of quantum operator algebra, not an arbitrary energy offset added by hand.

5.3 The number operator and energy eigenstates

Define the *number operator*

$$\hat{N} = \hat{a}^\dagger \hat{a}. \tag{54}$$

The harmonic-oscillator Hamiltonian can then be written as

$$\hat{H} = \hbar\omega \left(\hat{N} + \frac{1}{2} \right). \tag{55}$$

Because $\hat{H}$ is a linear function of $\hat{N}$, the two operators share the same eigenstates. Let $|n\rangle$ be an eigenstate of the number operator,

$$\hat{N} |n\rangle = n |n\rangle . \tag{56}$$

Acting on the same state with the Hamiltonian gives

$$\hat{H} |n\rangle = \hbar\omega \left(\hat{N} + \frac{1}{2} \right) |n\rangle \tag{57}$$

$$= \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle . \tag{58}$$

Thus, $|n\rangle$ is also an energy eigenstate, with energy

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right). \tag{59}$$

5.4 Hermiticity, real eigenvalues, and orthogonality

An observable is represented by a Hermitian operator. The Hamiltonian is Hermitian,

$$\hat{H}^\dagger = \hat{H}, \tag{60}$$

and the number operator is also Hermitian because

$$\hat{N}^\dagger = \left(\hat{a}^\dagger \hat{a} \right)^\dagger = \hat{a}^\dagger \hat{a} = \hat{N}. \tag{61}$$

Hermitian operators have real eigenvalues. Their eigenstates associated with distinct eigenvalues are orthogonal. Therefore,

$$n \neq m \quad \Longrightarrow \quad \langle m|n\rangle = 0. \tag{62}$$

The number states may consequently be chosen to form an orthonormal set,

$$\langle m|n\rangle = \delta_{mn}. \tag{63}$$

5.5 Allowed eigenvalues of the number operator

The eigenvalues of $\hat{N}$ cannot be negative. For an eigenstate $|n\rangle$,

$$n \langle n|n\rangle = \langle n| \hat{N} |n\rangle \tag{64}$$

$$= \langle n| \hat{a}^\dagger \hat{a} |n\rangle \tag{65}$$

$$= \| \hat{a} |n\rangle \|^2 \geq 0. \tag{66}$$

For a normalized state, $\langle n|n\rangle = 1$, and hence

$$n \geq 0. \tag{67}$$

At this stage, positivity establishes only that the eigenvalues are nonnegative. The ladder-operator construction below will show that the allowed values are the nonnegative integers,

$$n = 0, 1, 2, \ldots . \tag{68}$$

Once this discrete spectrum has been established, the corresponding energies are

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \qquad n = 0, 1, 2, \ldots . \tag{69}$$

In particular, the lowest allowed energy is

$$E_0 = \frac{1}{2} \hbar\omega, \tag{70}$$

which is the zero-point energy anticipated by the operator ordering above.

5.6 Action of the raising and lowering operators

We now determine how the operators $\hat{a}^\dagger$ and $\hat{a}$ act on a number-operator eigenstate. The useful commutators are

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger, \qquad [\hat{N}, \hat{a}] = -\hat{a}. \tag{71}$$

They follow directly from $\hat{N} = \hat{a}^\dagger \hat{a}$ and $[\hat{a}, \hat{a}^\dagger] = 1$. For example,

$$\begin{aligned} [\hat{N}, \hat{a}^\dagger] &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] && (72)\\ &= \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} && (73)\\ &= \hat{a}^\dagger, && (74) \end{aligned}$$

and similarly $[\hat{N}, \hat{a}] = -\hat{a}$.

5.7 Raising the number eigenvalue

Starting from

$$\hat{N} \left|n\right\rangle = n \left|n\right\rangle, \tag{75}$$

we act with $\hat{N}$ on the state $\hat{a}^\dagger \left|n\right\rangle$:

$$\begin{aligned} \hat{N}\hat{a}^\dagger \left|n\right\rangle &= \left(\hat{a}^\dagger \hat{N} + [\hat{N}, \hat{a}^\dagger]\right) \left|n\right\rangle && (76)\\ &= \hat{a}^\dagger n \left|n\right\rangle + \hat{a}^\dagger \left|n\right\rangle && (77)\\ &= (n+1)\hat{a}^\dagger \left|n\right\rangle. && (78) \end{aligned}$$

Thus, $\hat{a}^\dagger \left|n\right\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+1$. It must therefore be proportional to $\left|n+1\right\rangle$:

$$\hat{a}^\dagger \left|n\right\rangle = c_n \left|n+1\right\rangle. \tag{79}$$

The magnitude of the coefficient follows from the norm,

$$\begin{aligned} |c_n|^2 &= \left\|\hat{a}^\dagger \left|n\right\rangle\right\|^2 && (80)\\ &= \left\langle n\right| \hat{a}\hat{a}^\dagger \left|n\right\rangle && (81)\\ &= \left\langle n\right| (\hat{N}+1) \left|n\right\rangle && (82)\\ &= n+1. && (83) \end{aligned}$$

The overall phase of each number state is arbitrary, so we adopt the standard phase convention in which c_n is real and positive. Hence,

$$\boxed{\hat{a}^\dagger \left|n\right\rangle = \sqrt{n+1}\ \left|n+1\right\rangle}. \tag{84}$$

5.8 Lowering the number eigenvalue

Likewise,

$$\begin{aligned} \hat{N}\hat{a} \left|n\right\rangle &= \left(\hat{a}\hat{N} + [\hat{N}, \hat{a}]\right) \left|n\right\rangle && (85)\\ &= \hat{a}n \left|n\right\rangle - \hat{a} \left|n\right\rangle && (86)\\ &= (n-1)\hat{a} \left|n\right\rangle. && (87) \end{aligned}$$

Therefore, $\hat{a} \left|n\right\rangle$ is proportional to $\left|n-1\right\rangle$:

$$\hat{a} \left|n\right\rangle = d_n \left|n-1\right\rangle. \tag{88}$$

Again, the norm determines the magnitude,

$$\begin{aligned} |d_n|^2 &= \left\|\hat{a} \left|n\right\rangle\right\|^2 && (89)\\ &= \left\langle n\right| \hat{a}^\dagger \hat{a} \left|n\right\rangle && (90)\\ &= \left\langle n\right| \hat{N} \left|n\right\rangle && (91)\\ &= n. && (92) \end{aligned}$$

With the same phase convention,

$$\boxed{\hat{a}\,|n\rangle = \sqrt{n}\;|n-1\rangle}. \tag{93}$$

For the lowest state, $n = 0$, this result becomes

$$\hat{a}\,|0\rangle = 0. \tag{94}$$

The lowering sequence therefore terminates at $|0\rangle$, while repeated application of $\hat{a}^\dagger$ generates states with successively larger nonnegative integer eigenvalues. This establishes the discrete number spectrum and prepares the construction of all number states from the ground state.

5.9 Constructing the number-state basis

The ground state, or vacuum state, is defined by

$$\hat{a}\,|0\rangle = 0. \tag{95}$$

All higher number states can be generated by repeated application of the raising operator. From the result

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\;|n+1\rangle\,, \tag{96}$$

we first obtain

$$\hat{a}^\dagger\,|0\rangle = |1\rangle\,, \tag{97}$$

and then

$$\hat{a}^\dagger\,|1\rangle = \sqrt{2}\;|2\rangle\,. \tag{98}$$

Consequently,

$$(\hat{a}^\dagger)^2\,|0\rangle = \sqrt{2}\;|2\rangle\,, \tag{99}$$

so that

$$|2\rangle = \frac{1}{\sqrt{2!}}(\hat{a}^\dagger)^2\,|0\rangle\,. \tag{100}$$

Continuing in this way gives the normalized number state

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0\rangle} \qquad n = 0, 1, 2, \ldots. \tag{101}$$

The factorial normalization follows directly from repeated use of the raising relation:

$$(\hat{a}^\dagger)^n\,|0\rangle = \sqrt{1}\sqrt{2}\cdots\sqrt{n}\;|n\rangle \tag{102}$$

$$= \sqrt{n!}\;|n\rangle\,. \tag{103}$$

Equivalently, if $|n\rangle$ is normalized, then

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}^\dagger\,|n\rangle \tag{104}$$

is normalized as well, because

$$\langle n+1|n+1\rangle = \frac{1}{n+1}\,\langle n|\,\hat{a}\hat{a}^\dagger\,|n\rangle \tag{105}$$

$$= \frac{1}{n+1}\,\langle n|\,(\hat{N}+1)\,|n\rangle \tag{106}$$

$$= 1. \tag{107}$$

The complete ladder relations are therefore

$$\boxed{\hat{a}\,|n\rangle = \sqrt{n}\;|n-1\rangle\,, \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\;|n+1\rangle} \tag{108}$$

with $\hat{a}\,|0\rangle = 0$. The states $\{|n\rangle\}$ form the number-state, or Fock-state, basis for the harmonic oscillator.

Because

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right), \tag{109}$$

each number state is also an energy eigenstate:

$$\hat{H}\,|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle\,. \tag{110}$$

Thus,

$$\boxed{E_n = \hbar\omega\left(n+\frac{1}{2}\right)}, \qquad n = 0,1,2,\ldots. \tag{111}$$

In particular, the vacuum is not a zero-energy state:

$$\hat{H}\,|0\rangle = \frac{1}{2}\hbar\omega\,|0\rangle\,. \tag{112}$$

Its energy is the zero-point energy $E_0 = \hbar\omega/2$, which arises from the noncommutativity of the raising and lowering operators.

> **The oscillator ladder.** The vacuum $|0\rangle$ is the lowest number state, and repeated application of $\hat{a}^\dagger$ generates the complete Fock basis. The ladder structure simultaneously explains the integer spacing of number eigenvalues and the equally spaced oscillator energies.

6 What we learned

These notes introduced the basic language of quantum mechanics through state vectors, bras and kets, inner products, superposition, normalization, and the Born interpretation of measurement. The double-slit experiment showed that an individual trial produces one localized outcome, whereas an ensemble of identically prepared trials reveals a reproducible probability distribution. Interference occurs because coherent probability amplitudes are combined before taking the absolute square; this distinguishes a quantum superposition from a classical statistical mixture.

The quantum harmonic oscillator then demonstrated how operator algebra reveals physical structure. The Hamiltonian and number operator share eigenstates, the raising and lowering operators connect neighboring number states, and repeated application of the raising operator generates the complete Fock basis from the vacuum. The noncommutativity of the ladder operators produces the zero-point energy and fixes the equally spaced spectrum. These ideas provide the foundation for the quantum description of light developed in subsequent notes.

7 References

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