Quantum Photonics

Quantized Light and Linear Optical Transformations

Purpose: Introduce the quantum description of light and the linear-optical transformations used to manipulate photonic states, from single-mode Fock states and beam splitters to Hong–Ou–Mandel interference, programmable MZI meshes, and squeezed light.

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Prerequisites. Complex numbers, basic linear algebra, Dirac notation, superposition and measurement, and the quantum harmonic oscillator.

Learning objectives.

- connect optical modes with harmonic-oscillator Fock states;
- distinguish classical and quantum beam-splitter transformations;
- derive the Hong–Ou–Mandel output probabilities;
- explain how MZI meshes realize general linear-optical unitaries;
- relate squeezing to quadrature noise and mean photon number.

1 Introduction

Quantum photonics applies the state and operator language of quantum mechanics to electromagnetic fields. The central connection is that each independent optical mode behaves mathematically as a quantum harmonic oscillator. Its Fock states describe definite photon numbers, while creation and annihilation operators provide the natural language for optical transformations.

The beam splitter is the basic two-mode element. Classically, its unitary matrix conserves optical power. Quantum mechanically, the same mode transformation preserves total photon number but acts on creation and annihilation operators. Applying this transformation to a two-photon input reveals Hong–Ou–Mandel interference, where probability amplitudes cancel the coincidence outcome at a balanced beam splitter.

Phase shifters and beam splitters can be combined into tunable Mach–Zehnder interferometers and then assembled into universal meshes. The lecture closes with squeezed light, illustrating that important optical states need not have a definite photon number and may instead be characterized by reduced noise in one field quadrature.

2 Quantization of Light

2.1 Historical Motivation

Planck introduced discrete energy elements in his description of blackbody radiation, with oscillator energies separated by $\hbar\omega$ [1]:

$$E_n = n\hbar\omega, \qquad n = 0, 1, 2, \ldots. \tag{1}$$

Einstein subsequently interpreted light energy as localized quanta in his explanation of the photoelectric effect [2]. These historical steps established energy quantization as a fundamental property of light.

2.2 Connection to the Quantum Harmonic Oscillator

The quantum harmonic oscillator has the spectrum

$$E_n^{(\mathrm{QHO})} = \hbar\omega\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots. \tag{2}$$

Apart from the constant zero-point contribution $\hbar\omega/2$, both spectra have the same level spacing,

$$E_{n+1} - E_n = \hbar\omega. \tag{3}$$

A single mode of the electromagnetic field is therefore described mathematically as a quantum harmonic oscillator. Its number states are the photon-number, or Fock, states of that optical mode.

> **Physics insight.** A single optical mode is a quantum harmonic oscillator. Photon-number states are therefore the same number states introduced for the oscillator, now interpreted as occupations of one field mode.

3 Quantum Harmonic Oscillator Recap

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= 1, && (4)\\
\hat{N} &= \hat{a}^\dagger \hat{a}, && (5)\\
\hat{N}\,|n\rangle &= n\,|n\rangle\,, && (6)\\
\hat{H} &= \hbar\omega\left(\hat{N} + \frac{1}{2}\right), && (7)\\
\hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,, && (8)\\
\hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\,, && (9)\\
\hat{a}\,|0\rangle &= 0. && (10)
\end{aligned}$$

$$\boxed{|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle}, \qquad n = 0, 1, 2, \ldots. \tag{11}$$

> **Physics insight.** The harmonic-oscillator recap is not a separate formalism: it is the operator toolkit used throughout quantum optics. The state $|n\rangle$ contains exactly n photons in the selected mode.

4 Beam Splitters

A lossless beam splitter mixes two optical modes while conserving total optical power. Let the complex classical input and output field amplitudes be collected into

$$\mathbf{E}_{\text{in}} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \qquad \mathbf{E}_{\text{out}} = \begin{pmatrix} E_3 \\ E_4 \end{pmatrix}, \tag{12}$$

and write the linear transformation as

$$\mathbf{E}_{\text{out}} = U\mathbf{E}_{\text{in}}. \tag{13}$$

Conservation of total input and output power for every possible input requires

$$\begin{aligned}
\mathbf{E}_{\text{out}}^\dagger \mathbf{E}_{\text{out}} &= \mathbf{E}_{\text{in}}^\dagger \mathbf{E}_{\text{in}}, && (14)\\
\mathbf{E}_{\text{in}}^\dagger U^\dagger U \mathbf{E}_{\text{in}} &= \mathbf{E}_{\text{in}}^\dagger \mathbf{E}_{\text{in}}. && (15)
\end{aligned}$$

Because this relation must hold for arbitrary $\mathbf{E}_{\text{in}}$,

$$\boxed{U^\dagger U = I}. \tag{16}$$

Thus, a lossless two-mode optical transformation must be unitary.

A useful parameter-counting sanity check is

$$8 \xrightarrow{U^\dagger U = I} 4 \xrightarrow{\text{remove global phase}} 3 \xrightarrow{\text{choose a mode-phase convention}} 2. \tag{17}$$

A general 2×2 complex matrix contains eight real parameters; unitarity leaves four, the physically irrelevant global phase leaves three, and a fixed convention for the phases of the input and output modes leaves the two beam-splitter parameters used below. Appendix A gives the more detailed $U(2)$ and $SU(2)$ discussion.

4.1 Classical Beam Splitter in the Gerry–Knight Convention

Up to a global phase and with a fixed mode-phase convention, the Gerry–Knight beam-splitter transformation is [6]

$$U_{\mathrm{GK}}(\theta,\phi) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ -e^{-i\phi}\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{18}$$

The classical field amplitudes therefore transform according to

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = U_{\mathrm{GK}}(\theta,\phi)\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}. \tag{19}$$

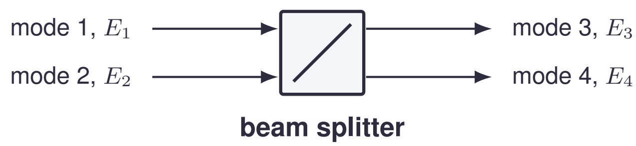

Figure 1: Input modes 1 and 2 and output modes 3 and 4 for the beam-splitter transformation.

The complex transmission and reflection amplitudes may be identified as

$$t = \cos\left(\frac{\theta}{2}\right), \qquad r = e^{i\phi}\sin\left(\frac{\theta}{2}\right), \tag{20}$$

with the lower-left matrix element fixed by unitarity to $-r^*$. The corresponding power coefficients are

$$T = |t|^2 = \cos^2\left(\frac{\theta}{2}\right), \qquad R = |r|^2 = \sin^2\left(\frac{\theta}{2}\right), \tag{21}$$

so that $T + R = 1$.

For a balanced beam splitter,

$$\theta = \frac{\pi}{2}, \qquad T = R = \frac{1}{2}. \tag{22}$$

The phase ϕ changes the numerical matrix elements. For example,

$$U_{\mathrm{GK}}\left(\frac{\pi}{2},0\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{23}$$

$$U_{\mathrm{GK}}\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{24}$$

These two matrices are examples of the same general Gerry–Knight parameterization. We will retain the full ϕ-dependence in the quantum treatment and show that the Hong–Ou–Mandel output probabilities are independent of ϕ.

Power conservation can be checked explicitly. For the classical input

$$\mathbf{E}_{\mathrm{in}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad P_{\mathrm{in}} = |1|^2 + |1|^2 = 2, \tag{25}$$

the balanced transformation gives

$$\mathbf{E}_{\mathrm{out}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 + e^{i\phi} \\ 1 - e^{-i\phi} \end{pmatrix}. \tag{26}$$

Hence,

$$\begin{aligned} P_{\mathrm{out}} &= \frac{1}{2}\left|1 + e^{i\phi}\right|^2 + \frac{1}{2}\left|1 - e^{-i\phi}\right|^2 && (27)\\ &= \frac{1}{2}(2 + 2\cos\phi) + \frac{1}{2}(2 - 2\cos\phi) && (28)\\ &= 2 = P_{\mathrm{in}}. && (29) \end{aligned}$$

For $\phi = 0$, all classical power exits through one output port. For $\phi = \pi/2$, however,

$$\mathbf{E}_{\text{out}} = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{30}$$

Thus, apart from a global phase, the balanced classical input vector $(1,1)^{\mathsf{T}}$ is unchanged, while total power is conserved in both examples.

This result exposes the limit of the classical description. If the vector $(1,1)^{\mathsf{T}}$ were naively interpreted as the photon-number state $|1,1\rangle$, the calculation above would suggest one photon in each output mode. The Hong–Ou–Mandel experiment instead gives zero coincidence probability for two indistinguishable photons at a balanced beam splitter [7]. We are therefore missing an essential step: the unitary matrix acts on classical field amplitudes in the calculation above, but quantum photonics requires the corresponding transformation of creation and annihilation operators. The next subsection develops that operator description.

4.2 Quantum Beam Splitter

For quantum light, the same unitary matrix transforms the mode operators rather than photon-number labels. Introduce the annihilation-operator columns

$$\hat{\mathbf{a}}_{\text{in}} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \qquad \hat{\mathbf{a}}_{\text{out}} = \begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix}. \tag{31}$$

The Gerry–Knight transformation is

$$\hat{\mathbf{a}}_{\text{out}} = U_{\text{GK}}(\theta,\phi)\hat{\mathbf{a}}_{\text{in}}, \tag{32}$$

or explicitly,

$$\hat{a}_3 = \cos\left(\frac{\theta}{2}\right)\hat{a}_1 + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_2, \tag{33}$$

$$\hat{a}_4 = -e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_1 + \cos\left(\frac{\theta}{2}\right)\hat{a}_2. \tag{34}$$

Taking the Hermitian conjugate gives the corresponding creation-operator relations,

$$\hat{a}_3^\dagger = \cos\left(\frac{\theta}{2}\right)\hat{a}_1^\dagger + e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_2^\dagger, \tag{35}$$

$$\hat{a}_4^\dagger = -e^{i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_1^\dagger + \cos\left(\frac{\theta}{2}\right)\hat{a}_2^\dagger. \tag{36}$$

Unitarity preserves the bosonic commutation relations. In compact notation,

$$[\hat{a}_{\text{out},j}, \hat{a}_{\text{out},k}^\dagger] = \sum_{m,n} U_{jm}U_{kn}^*[\hat{a}_{\text{in},m}, \hat{a}_{\text{in},n}^\dagger] \tag{37}$$

$$= \sum_{m} U_{jm}U_{km}^* = (UU^\dagger)_{jk} = \delta_{jk}. \tag{38}$$

Thus, the output operators describe valid bosonic modes whenever the optical transformation is unitary.

For a passive, lossless beam splitter, unitarity also conserves the total photon number. Define

$$\hat{N}_{\text{in}} = \hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2, \qquad \hat{N}_{\text{out}} = \hat{a}_3^\dagger\hat{a}_3 + \hat{a}_4^\dagger\hat{a}_4. \tag{39}$$

Using the operator columns,

$$\hat{N}_{\text{out}} = \hat{\mathbf{a}}_{\text{out}}^\dagger\hat{\mathbf{a}}_{\text{out}} \tag{40}$$

$$= \hat{\mathbf{a}}_{\text{in}}^\dagger U_{\text{GK}}^\dagger U_{\text{GK}}\hat{\mathbf{a}}_{\text{in}} \tag{41}$$

$$= \hat{\mathbf{a}}_{\text{in}}^\dagger\hat{\mathbf{a}}_{\text{in}} \tag{42}$$

$$= \hat{N}_{\text{in}}. \tag{43}$$

The beam splitter may redistribute photons between its two output modes, but it cannot create or destroy photons: the sum of the mode occupations is unchanged. This is the quantum counterpart of classical power conservation.

The operator transformation now provides the missing link between the optical unitary and quantum states. The next section applies it to the two-photon input $|1,1\rangle$. That calculation produces the Hong–Ou–Mandel effect; the full state transformation and its output probabilities are deferred to that dedicated section.

Physics insight. The same unitary matrix appears in classical and quantum optics, but it acts on different objects. Classical field amplitudes transform directly; photon-number states must be transformed through the creation and annihilation operators.

5 Hong–Ou–Mandel Interference

5.1 Two-Photon Input and Allowed Outputs

Consider one photon in each input mode,

$$|1,1\rangle_{12} = \hat{a}_1^\dagger \hat{a}_2^\dagger |0,0\rangle_{12}. \tag{44}$$

The beam splitter is passive and lossless, so the total photon number remains $N = 2$. The output state must therefore lie in the three-dimensional number-conserving subspace

$$\boxed{|2,0\rangle_{34}, \qquad |1,1\rangle_{34}, \qquad |0,2\rangle_{34}}. \tag{45}$$

This is already a fundamental departure from the classical field-amplitude calculation. A classical input vector is mapped to one classical output vector. A two-photon quantum input can instead become a coherent superposition of all output Fock states compatible with total photon-number conservation.

To transform the input state, the input creation operators must be written in terms of the output creation operators. From

$$\hat{\mathbf{a}}_{\text{out}} = U_{\text{GK}}(\theta,\phi)\hat{\mathbf{a}}_{\text{in}}, \tag{46}$$

unitarity gives

$$\hat{\mathbf{a}}_{\text{in}} = U_{\text{GK}}^\dagger(\theta,\phi)\hat{\mathbf{a}}_{\text{out}}. \tag{47}$$

Thus,

$$\hat{a}_1 = \cos\left(\frac{\theta}{2}\right)\hat{a}_3 - e^{i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_4, \tag{48}$$

$$\hat{a}_2 = e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_3 + \cos\left(\frac{\theta}{2}\right)\hat{a}_4. \tag{49}$$

Taking Hermitian conjugates,

$$\hat{a}_1^\dagger = \cos\left(\frac{\theta}{2}\right)\hat{a}_3^\dagger - e^{-i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_4^\dagger, \tag{50}$$

$$\hat{a}_2^\dagger = e^{i\phi}\sin\left(\frac{\theta}{2}\right)\hat{a}_3^\dagger + \cos\left(\frac{\theta}{2}\right)\hat{a}_4^\dagger. \tag{51}$$

For compactness, define

$$c = \cos\left(\frac{\theta}{2}\right), \qquad s = \sin\left(\frac{\theta}{2}\right). \tag{52}$$

The two-photon input then transforms as

$$\begin{aligned}
|\psi_{\text{out}}\rangle &= \hat{a}_1^\dagger \hat{a}_2^\dagger |0,0\rangle_{34} && (53)\\
&= \left(c\hat{a}_3^\dagger - e^{-i\phi}s\hat{a}_4^\dagger\right)\left(e^{i\phi}s\hat{a}_3^\dagger + c\hat{a}_4^\dagger\right)|0,0\rangle_{34} && (54)\\
&= \left[cse^{i\phi}(\hat{a}_3^\dagger)^2 + c^2\hat{a}_3^\dagger\hat{a}_4^\dagger - s^2\hat{a}_4^\dagger\hat{a}_3^\dagger - cse^{-i\phi}(\hat{a}_4^\dagger)^2\right]|0,0\rangle_{34}. && (55)
\end{aligned}$$

Operators belonging to different output modes commute,

$$[\hat{a}_3^\dagger, \hat{a}_4^\dagger] = 0, \tag{56}$$

so the two middle terms combine:

$$|\psi_{\text{out}}\rangle = \left[cse^{i\phi}(\hat{a}_3^\dagger)^2 + (c^2 - s^2)\hat{a}_3^\dagger\hat{a}_4^\dagger - cse^{-i\phi}(\hat{a}_4^\dagger)^2\right]|0,0\rangle_{34}. \tag{57}$$

Using

$$(\hat{a}_3^\dagger)^2 |0,0\rangle_{34} = \sqrt{2}\, |2,0\rangle_{34}\,, \tag{58}$$
$$\hat{a}_3^\dagger \hat{a}_4^\dagger |0,0\rangle_{34} = |1,1\rangle_{34}\,, \tag{59}$$
$$(\hat{a}_4^\dagger)^2 |0,0\rangle_{34} = \sqrt{2}\, |0,2\rangle_{34}\,, \tag{60}$$

and the identities

$$2cs = \sin\theta, \qquad c^2 - s^2 = \cos\theta, \tag{61}$$

we obtain the normalized output state

$$\boxed{|\psi_{\text{out}}\rangle = \frac{e^{i\phi}\sin\theta}{\sqrt{2}}\, |2,0\rangle_{34} + \cos\theta\, |1,1\rangle_{34} - \frac{e^{-i\phi}\sin\theta}{\sqrt{2}}\, |0,2\rangle_{34}}. \tag{62}$$

5.2 Output Probabilities

The three mutually orthogonal output states correspond to three distinct photon-counting outcomes. Their probabilities are the absolute squares of the associated amplitudes:

$$P_{20} = \left|\frac{e^{i\phi}\sin\theta}{\sqrt{2}}\right|^2 = \frac{1}{2}\sin^2\theta, \tag{63}$$
$$P_{11} = |\cos\theta|^2 = \cos^2\theta, \tag{64}$$
$$P_{02} = \left|-\frac{e^{-i\phi}\sin\theta}{\sqrt{2}}\right|^2 = \frac{1}{2}\sin^2\theta. \tag{65}$$

The phase ϕ remains in the quantum amplitudes but cancels from every photon-counting probability because $|e^{\pm i\phi}|^2 = 1$. The probabilities are correctly normalized:

$$P_{20} + P_{11} + P_{02} = \frac{1}{2}\sin^2\theta + \cos^2\theta + \frac{1}{2}\sin^2\theta = 1. \tag{66}$$

For a balanced beam splitter, $\theta = \pi/2$, and therefore

$$P_{11} = 0, \qquad P_{20} = P_{02} = \frac{1}{2}. \tag{67}$$

The output state becomes

$$\boxed{|\psi_{\text{out}}\rangle^{\text{bal}} = \frac{1}{\sqrt{2}}\left(e^{i\phi}\,|2,0\rangle_{34} - e^{-i\phi}\,|0,2\rangle_{34}\right)}. \tag{68}$$

The photons always leave together through one output port or the other. The coincidence outcome $|1,1\rangle_{34}$ is absent because its two indistinguishable probability amplitudes cancel.

5.3 Coincidence Dip

The coincidence probability is

$$P_{11}(\theta) = \cos^2\theta. \tag{69}$$

It vanishes at the balanced setting $\theta = \pi/2 = 90^\circ$, producing the Hong–Ou–Mandel dip [7].

> **Physics insight.** Hong–Ou–Mandel interference is two-photon amplitude interference. At a balanced beam splitter the two alternatives leading to one photon in each output cancel, while the two bunched outcomes remain.

6 Phase Shifters

A phase shifter changes the phase of one optical mode without mixing the two modes. For a phase shift applied to the second mode, the two-mode transformation is

$$U_{\text{PS}}(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}. \tag{70}$$

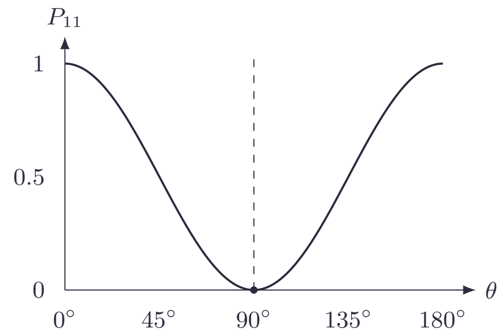

Figure 2: Coincidence probability for the two-photon input $|1,1\rangle$. The ideal Hong–Ou–Mandel dip reaches zero at the balanced setting $\theta = 90^\circ$.

The matrix is unitary,

$$U_{\mathrm{PS}}^{\dagger}(\varphi)U_{\mathrm{PS}}(\varphi) = I, \tag{71}$$

and therefore preserves total optical power in the classical description and total photon number in the quantum description. It changes only the phase of the second mode:

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = U_{\mathrm{PS}}(\varphi) \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} \hat{a}_1 \\ e^{i\varphi}\hat{a}_2 \end{pmatrix}. \tag{72}$$

A phase shift on the first mode would instead be represented by $\mathrm{diag}(e^{i\varphi}, 1)$.

Two sequential phase shifters acting on the same mode segment combine into a single effective phase:

$$\begin{aligned} U_{\mathrm{PS}}(\varphi_2)U_{\mathrm{PS}}(\varphi_1) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_1} \end{pmatrix} && (73) \\ &= U_{\mathrm{PS}}(\varphi_1 + \varphi_2). && (74) \end{aligned}$$

Their phases add modulo 2π, so consecutive phase shifters on an uninterrupted mode path do not provide independent degrees of freedom.

6.1 Mach–Zehnder Interferometer as a Tunable Two-Mode Gadget

A Mach–Zehnder interferometer combines two fixed beam splitters with a phase shifter in one internal arm,

$$\mathrm{BS} \;\longrightarrow\; \mathrm{PS}(\varphi) \;\longrightarrow\; \mathrm{BS}. \tag{75}$$

Once the two beam splitters are fixed, the internal phase φ is the only tunable degree of freedom. The device is therefore a one-parameter two-mode transformation rather than a general element of $U(2)$.

For two identically oriented balanced beam splitters in the Gerry–Knight convention,

$$B = U_{\mathrm{GK}}\left(\frac{\pi}{2}, 0\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{76}$$

and for a phase shift on the second internal mode,

$$U_{\mathrm{MZI}}(\varphi) = B\, U_{\mathrm{PS}}(\varphi)\, B. \tag{77}$$

Carrying out the matrix multiplication gives

$$\begin{aligned} U_{\mathrm{MZI}}(\varphi) &= \frac{1}{2} \begin{pmatrix} 1 - e^{i\varphi} & 1 + e^{i\varphi} \\ -\left(1 + e^{i\varphi}\right) & -1 + e^{i\varphi} \end{pmatrix} && (78) \\ &= e^{i\varphi/2} \begin{pmatrix} -i\sin(\varphi/2) & \cos(\varphi/2) \\ -\cos(\varphi/2) & i\sin(\varphi/2) \end{pmatrix}. && (79) \end{aligned}$$

The factor $e^{i\varphi/2}$ is a global phase. Up to that phase, varying the single internal phase continuously changes the effective mixing between the two modes. Two simple settings illustrate the routing behavior:

$$U_{\mathrm{MZI}}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{80}$$

$$U_{\mathrm{MZI}}(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{81}$$

Thus, a fixed beam-splitter pair together with one controllable phase forms a compact tunable building block for larger linear-optical networks.

7 Mach–Zehnder Interferometer Meshes

7.1 Degrees of Freedom of a General Multimode Unitary

A passive lossless network acting on N optical modes is described by

$$\hat{\mathbf{a}}_{\mathrm{out}} = U\hat{\mathbf{a}}_{\mathrm{in}}, \qquad U \in U(N), \tag{82}$$

where

$$\hat{\mathbf{a}} = \begin{pmatrix} \hat{a}_1 & \hat{a}_2 & \cdots & \hat{a}_N \end{pmatrix}^{\mathsf{T}}. \tag{83}$$

An $N \times N$ complex matrix begins with $2N^2$ real parameters. The unitarity condition

$$U^\dagger U = I \tag{84}$$

imposes N^2 independent real constraints. Therefore,

$$\dim U(N) = 2N^2 - N^2 = N^2. \tag{85}$$

A universal interferometer must consequently provide N^2 independently tunable real degrees of freedom.

7.2 A Two-Parameter Primitive

The one-parameter MZI introduced above becomes a two-parameter primitive when a phase shifter is added after the second beam splitter:

$$\mathrm{BS} \longrightarrow \mathrm{PS}(\varphi) \longrightarrow \mathrm{BS} \longrightarrow \mathrm{PS}(\gamma). \tag{86}$$

Equivalently, this primitive is an MZI followed by a post-phase shifter. For fixed balanced beam splitters, the internal phase φ controls the mixing between the two modes, while the post-phase γ controls one additional relative output phase. The primitive therefore supplies two independent real degrees of freedom.

When embedded into a larger network, the primitive acts nontrivially only on a selected pair of modes j and k. Its 2×2 unitary block is inserted into the corresponding rows and columns of the $N \times N$ identity matrix, while all other modes pass through unchanged.

7.3 Clements Decomposition and Parameter Completion

A rectangular Clements mesh uses

$$N_{\mathrm{prim}} = \frac{N(N-1)}{2} \tag{87}$$

two-mode primitives to realize an arbitrary N-mode unitary [8]. Since each primitive supplies two tunable parameters, the interior mesh contributes

$$N_{\mathrm{dof,int}} = 2N_{\mathrm{prim}} = N(N-1) \tag{88}$$

real degrees of freedom.

The remaining N parameters are supplied by a diagonal boundary phase matrix,

$$D = \mathrm{diag}\left(e^{i\gamma_1}, e^{i\gamma_2}, \ldots, e^{i\gamma_N}\right). \tag{89}$$

The complete parameter count is therefore

$$\underbrace{N(N-1)}_{\text{two-mode primitives}} + \underbrace{N}_{\text{boundary phases}} = N^2, \tag{90}$$

which exactly matches $\dim U(N)$. In the Clements decomposition, these boundary phases appear as the final diagonal phase matrix. They are required to complete the general unitary transformation; they are not optional corrections to the mesh interior.

Schematically, a universal decomposition may be written as

$$U = D \prod_{\ell=1}^{N(N-1)/2} T_\ell, \tag{91}$$

where each T_ℓ is a two-mode MZI-plus-post-phase primitive embedded in the N-mode space. The order of the factors is fixed by the chosen mesh architecture and decomposition algorithm.

The Hong–Ou–Mandel calculation is the smallest nontrivial example of this general multimode action. In a larger interferometer, the same unitary mode transformation redistributes a fixed total photon number among many number-conserving Fock states, and the corresponding probability amplitudes interfere before detection probabilities are formed.

The placement of the remaining phase degrees of freedom is not unique. A particular decomposition may collect them in a final output phase bank, as in the standard Clements form, place an equivalent phase bank at the inputs, or distribute phase shifts among elements inside the mesh after a corresponding reparameterization. These alternatives implement the same underlying unitary physics. In a photonic integrated circuit, the preferred placement is therefore likely to be dictated by fabrication, routing, loss, thermal-control, and calibration constraints rather than by a fundamental physical requirement that the phases appear at one specific boundary.

Any redistributed phases must nevertheless remain independently addressable. Sequential phase shifters placed on the same uninterrupted mode segment are not independent: they collapse to one effective phase equal to their sum. Consequently, universality must be assessed by the number of independent parameters (or equivalently by the rank of the parameterization), not simply by counting physical phase-shifter elements. Practical phase placement may be chosen for fabrication convenience only after this independence condition is satisfied.

> **Physics insight.** Universality is controlled by independent degrees of freedom, not by the raw number of phase shifters. Sequential phases on the same uninterrupted mode combine into one parameter and cannot increase the rank of the mesh parameterization.

8 Squeezed Light

A photon-number state $|n\rangle$ has a definite integer eigenvalue of the number operator. More general quantum states of light need not have a definite photon number. Squeezed vacuum is an important example: it is a coherent superposition of Fock states and is most naturally characterized through its field quadratures.

8.1 Squeezed Vacuum and Photon Number

For a single optical mode, the squeezed-vacuum state is defined by

$$|\zeta\rangle = \hat{S}(\zeta)\,|0\rangle\,, \tag{92}$$

where

$$\hat{S}(\zeta) = \exp\left[\frac{1}{2}\left(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2}\right)\right], \qquad \zeta = re^{i\vartheta}. \tag{93}$$

Its expansion in the Fock basis is [6]

$$|\zeta\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \left(-e^{i\vartheta}\tanh r\right)^n |2n\rangle\,. \tag{94}$$

Only even photon-number states occur, but the state does not possess one fixed photon number. Its mean photon number is

$$\boxed{\langle\hat{N}\rangle = \sinh^2 r}. \tag{95}$$

Thus, a superposition state is generally described by a photon-number probability distribution and expectation values rather than by a single integer photon number.

8.2 Field Quadratures

Define the dimensionless field quadratures

$$\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \qquad \hat{P} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}. \tag{96}$$

They satisfy

$$[\hat{X}, \hat{P}] = i, \tag{97}$$

and the vacuum variances are

$$(\Delta X)^2_{\text{vac}} = (\Delta P)^2_{\text{vac}} = \frac{1}{2}. \tag{98}$$

For squeezing aligned with the X quadrature,

$$\boxed{(\Delta X)^2 = \frac{1}{2}e^{-2r}, \qquad (\Delta P)^2 = \frac{1}{2}e^{2r}} \tag{99}$$

so that

$$\Delta X\, \Delta P = \frac{1}{2} \tag{100}$$

for an ideal pure squeezed-vacuum state. One quadrature therefore has less noise than the vacuum, while the conjugate quadrature has correspondingly more noise. A nonzero squeezing phase ϑ rotates the squeezed and anti-squeezed axes in phase space.

8.3 Squeezing in Decibels

Experimental squeezing is commonly quoted in decibels relative to the vacuum variance. Taking positive S_{dB} to denote noise reduction,

$$S_{\text{dB}} = -10 \log_{10}\left[\frac{(\Delta X)^2}{(\Delta X)^2_{\text{vac}}}\right]. \tag{101}$$

Because the variance ratio is e^{-2r},

$$S_{\text{dB}} = \frac{20}{\ln 10}\, r \approx 8.686\, r. \tag{102}$$

Equivalently,

$$\boxed{r = \frac{\ln 10}{20} S_{\text{dB}} \approx 0.1151\, S_{\text{dB}}}. \tag{103}$$

For example,

$$3\ \text{dB} \iff r \approx 0.345, \qquad 10\ \text{dB} \iff r \approx 1.151. \tag{104}$$

Physics insight. Squeezed vacuum has no definite photon number: it is a coherent superposition of even Fock states. Squeezing reduces the variance of one quadrature below the vacuum level while increasing the conjugate quadrature variance.

9 What We Learned

A quantized optical mode inherits the Fock-state and ladder-operator structure of the quantum harmonic oscillator. Lossless beam splitters and phase shifters act through unitary mode transformations, conserving classical power and, in the quantum description, total photon number. The two-photon beam-splitter calculation showed how coherent amplitudes populate all allowed number-conserving outputs and produce the Hong–Ou–Mandel coincidence dip. Combining fixed beam splitters with tunable phases creates Mach–Zehnder primitives, and arranging these primitives in a Clements mesh supplies the independent parameters required for a general $U(N)$ transformation. The final squeezed-light example broadened the discussion beyond definite-number states by relating a Fock-state superposition to quadrature noise and the dimensionless squeezing parameter r.

A General Two-Mode Unitary Transformations

A general 2×2 complex matrix contains four complex entries, or eight real parameters. The condition

$$U^\dagger U = I \tag{105}$$

imposes four independent real constraints, leaving four real degrees of freedom. This is the dimension of $U(2)$.

Every $U \in U(2)$ can be written as

$$U = e^{i\chi} V, \qquad V \in SU(2), \tag{106}$$

where $\det V = 1$. The factor $e^{i\chi}$ multiplies both output amplitudes by the same phase. It does not alter classical output powers or quantum photon-counting probabilities, so the physically relevant mixing can be represented by $SU(2)$. Removing this global phase reduces the parameter count from four to three.

One convenient form of a general $SU(2)$ matrix is

$$V = \begin{pmatrix} e^{i\alpha}\cos\left(\dfrac{\theta}{2}\right) & e^{i\beta}\sin\left(\dfrac{\theta}{2}\right) \\ -e^{-i\beta}\sin\left(\dfrac{\theta}{2}\right) & e^{-i\alpha}\cos\left(\dfrac{\theta}{2}\right) \end{pmatrix}, \tag{107}$$

which contains the three real parameters θ, α, and β. A beam-splitter convention fixes one remaining phase freedom by selecting definite phase references for the input and output modes. With the Gerry–Knight phase convention, the resulting two-parameter form is

$$U_{\mathrm{GK}}(\theta, \phi) = \begin{pmatrix} \cos\left(\dfrac{\theta}{2}\right) & e^{i\phi}\sin\left(\dfrac{\theta}{2}\right) \\ -e^{-i\phi}\sin\left(\dfrac{\theta}{2}\right) & \cos\left(\dfrac{\theta}{2}\right) \end{pmatrix}. \tag{108}$$

The two remaining parameters control the mixing ratio and the relative phase. The choice of mode-phase convention changes matrix phases but not invariant output probabilities.

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